CLOSED INCOMPRESSIBLE SURFACES
IN KNOT COMPLEMENTS

ELIZABETH FINKELSTEIN AND YOAV MORIAH

ABSTRACT. In this paper we show that given a knot or link K in a $2n$-plat projection with $n \geq 3$ and $m \geq 5$, where m is the length of the plat, if the twist coefficients $a_{i,j}$ all satisfy $|a_{i,j}| > 1$ then $S^3 - N(K)$ has at least $2n - 4$ nonisotopic essential meridional planar surfaces. In particular if K is a knot then $S^3 - N(K)$ contains closed incompressible surfaces. In this case the closed surfaces remain incompressible after all surgeries except perhaps along a ray of surgery coefficients in $\mathbb{Z} \oplus \mathbb{Z}$.

1. INTRODUCTION

Incompressible surfaces play a major role in the study of 3-manifolds. In the case of manifolds with nonempty boundary, the existence of incompressible surfaces with boundary is a well-known result (see [He]). However, deciding whether a 3-manifold, closed or with boundary, contains a closed incompressible surface not parallel to the boundary turned out to be a hard problem.

An incompressible and boundary incompressible surface not parallel to the boundary will be called an *essential* surface. A surface in $S^3 - N(K)$ with nonempty boundary will be called *meridional* if its boundary components are meridians for K.

Recall that all knots and links $K \subset S^3$ are known to have $2n$-plat projections for some n (see Definition 1.6 and [BZ]). Our main theorem is:

Theorem 1.1. *Let $K \subset S^3$ be a knot or a link in a $2n$-plat projection, with $n \geq 3$ and $m \geq 5$. If $|a_{i,j}| > 1$, for each i and j, then $S^3 - N(K)$ contains at least $2n - 4$ nonisotopic essential meridional planar surfaces.*

As immediate corollaries we obtain

Theorem 1.2. *Let $K \subset S^3$ be a knot admitting a $2n$-plat projection, with $n \geq 3$ and $m \geq 5$. If $|a_{i,j}| > 1$, for each i and j, then $S^3 - N(K)$ contains a closed essential surface.*

Proof. It follows from Theorem 2.0.3 of [CGLS] that if K is a knot and if $S^3 - N(K)$ contains an incompressible planar meridional surface, then $S^3 - N(K)$ contains a closed essential surface (see also Theorem 6.1 of [GR]). □

Theorem 1.3. *Let $K \subset S^3$ be a knot in a $2n$-plat projection, $n \geq 3$ and $m \geq 5$. If $|a_{i,j}| > 1$, for each i and j, then the knot K is a nontrivial knot.*
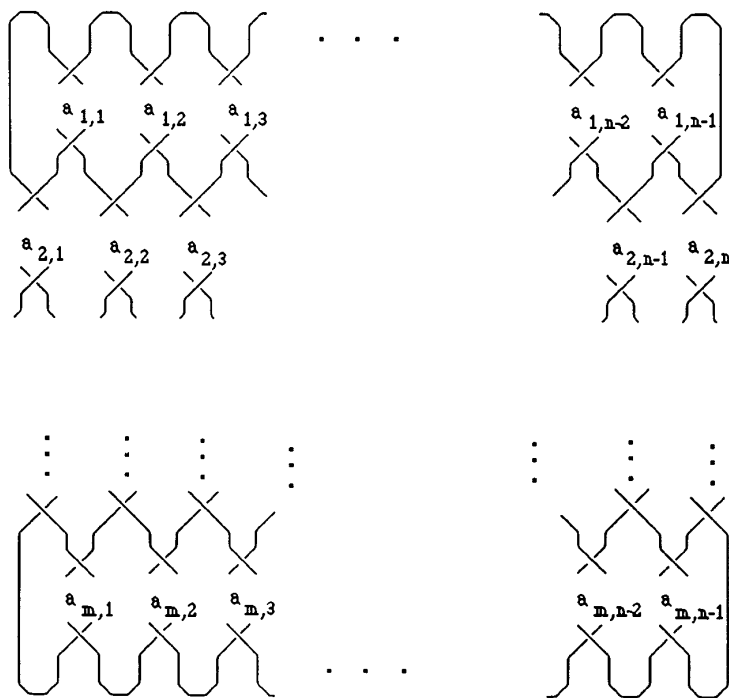
FIGURE 1

Remark 1.4. The condition that $|a_{i,j}| > 1$ can be weakened; this is done in Lemma 5.1 of Section 5. However, the statement we obtain there is probably still not the best possible.

Remark 1.5. Theorem 1.2 is applicable to knots only since there are examples of links, for example the Boromean rings, which contain essential planar meridional surfaces but do not contain closed incompressible surfaces.

Definition 1.6. A knot/link $K \subset S^3$ has a $2n$-plat projection if it can be projected onto a $2n$-braid in some plane with "bridges" connecting the strings on the top and on the bottom, so that the number of rows of crossings, m, is odd, as indicated in Figure 1 (see also [BZ]). Each configuration of the form
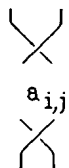
or
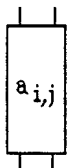
occurring in the ith row and the jth column, indicates $|a_{i,j}|$ crossings which are positive or negative depending on the sign of $a_{i,j}$, where $1 \leq i \leq m$ and $1 \leq j \leq n-1$ or $1 \leq j \leq n$, depending on whether the row is odd or even numbered respectively.

Remark 1.7. The fact that link complements contain essential surfaces is not that surprising since one can conceivably arrange the components of the link to be on either side of the surface in such a way that the surface is incompressible. However, the fact that essential surfaces are so abundant in knot complements is quite remarkable.

It follows from [Wu 1] that if the knot is not isotopic onto an incompressible surface, these surfaces will remain incompressible for all surgery coefficients in the surgery space, except perhaps for surgery coefficients on a single line. Let $\Delta(\gamma, \delta)$ denote the geometric intersection between geodesic curves on $\partial N(K)$ representing the surgery slopes γ and δ. We now have

Theorem 1.8. *Let $K \subset S^3$ be a knot in a $2n$-plat projection, $n \geq 3$, $m \geq 5$, with meridian μ. If $|a_{i,j}| > 1$, for each i and j, and K is not isotopic onto the essential surface, then all manifolds obtained by surgery, except surgery along slopes γ such that $\Delta(\gamma, \mu) \leq 1$, are Haken.* □

Remark 1.9. It is conjectured by Lopez that every non-Haken 3-manifold contains a knot whose complement contains no closed incompressible surface. The Lopez conjecture has implications to some approaches toward solving the Poincare Conjecture. Theorem 1.2, which deals only with knots in S^3, explains why the Lopez conjecture is hard to prove.

The known results about closed incompressible surfaces have been of three types. The first type of results consist of various specific methods of constructing manifolds which contain essential surfaces and hence tend to be of very special nature (see for example [Sw]). The second type assumes that essential surfaces exist in the 3-manifold and then it is shown that these surfaces must have certain properties (see [Sh], [Me]). The third type of results are actual existence theorems and there are surprisingly few of them. The only families of knots or links whose complements are known to contain essential surfaces, beside obvious constructions like satellite knots, are fibered knots (see [Ly]), Star links, (which include Montesinos knot/links), (see [Oe]), closed hyperbolic 3-braids (see [LP], [Fi]), Brunnian rings of at least four components, and homology boundary links (see [CL]).

2. 6-PLATS

In this section we will discuss knots or links which have projections as 6-plats (see for example Figure 2). This will be a fundamental case in our argument.

Definition 2.1. An n-tangle, $n > 1$, is a pair (B, T) where B is a 3-ball and T is a collection of n disjoint arcs (strings) $t_1, \ldots, t_n$ properly embedded in B. A tangle will be called essential if the planar surface $\partial B - N(\partial T)$ is incompressible in $B - N(T)$.

Definition 2.2. Let K be a knot or link in a 6-plat projection. A vertical 2-sphere in (S^3, K) is a 2-sphere intersecting $N(K)$ in meridional curves separating (S^3, K) into two tangles (B_1, T_1), (B_2, T_2). Furthermore, the 2-sphere intersects the projection plane of the plat in a unique simple closed curve. This curve intersects
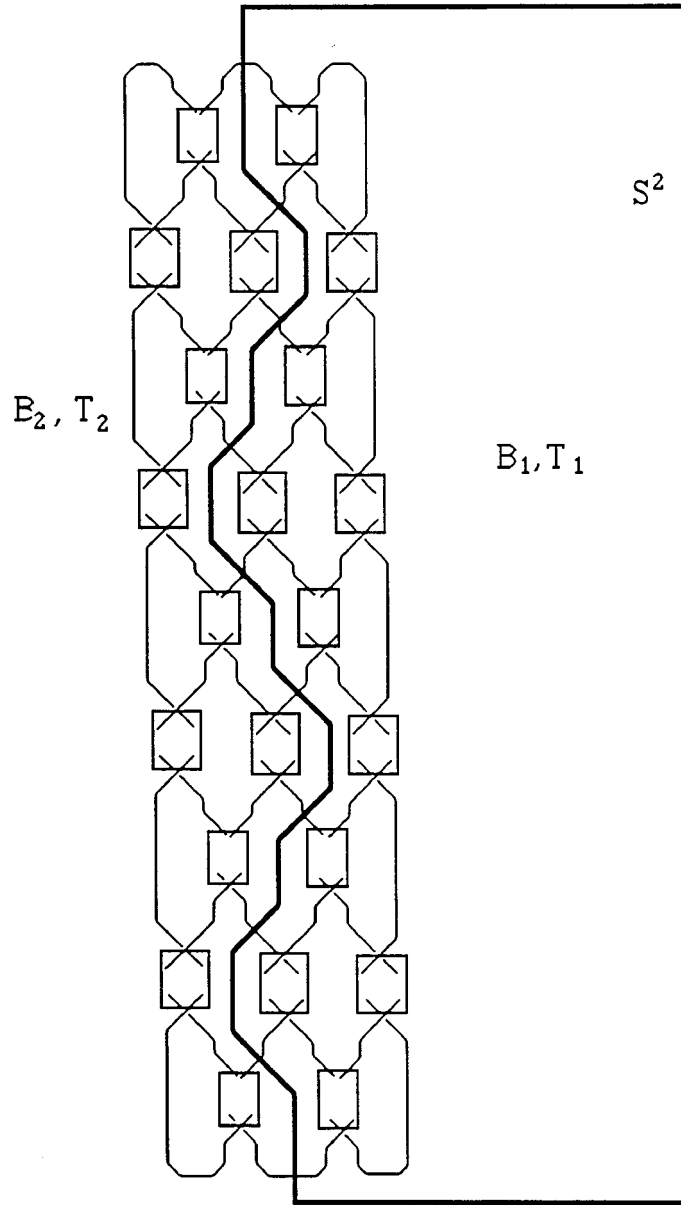
FIGURE 2

the top and bottom bridges and it separates an odd numbered row so that there is a box of crossings in each tangle and separates an even numbered row of three boxes of crossings into two boxes in one tangle and one box in the other alternately. The intersection of the 2-sphere with the projection plane is indicated by S^2 in Figure 2, for $m = 9, n = 3$.

Remark 2.3. Note that the definition of a vertical 2-sphere gives two such spheres al when $n = 3$, since the vertical sphere is determined by how it divides the first

row of three boxes into a set of two boxes and a single box and there are two such choices. In general we can define $2n - 4$ vertical 2-spheres in a $2n$-plat.

Lemma 2.4. *Let K be a knot or link in a 6-plat projection with $m \geq 5$. Then for each tangle (B_k, T_k), $k = 1, 2$, determined by a vertical 2-sphere, there is a collection of properly embedded disks $d_1, \ldots, d_r$, each of which intersects T_k in exactly two points. Furthermore, the disks $d_1, \ldots, d_r$ cut (B_k, T_k), $k = 1, 2$, into sub-tangles, each of which is homeomorphic, fixing the disks, to one of the seven sub-tangles as shown in Figure 3, up to rotation and reflection.*

Remark 2.5. Note that the tangles of type A^*, B^*, C^*, C^{**} are isotopic to the tangles A, B, C, respectively, by an isotopy which fixes the disk d_i (or disks in the case of tangles of type C).

Proof. Consider the vertical 2-sphere S cutting the knot or link in the 6-plat configuration into the two tangles. It is clear that each time S separates a row of three boxes of crossings into two boxes in one tangle and one box in the other, we can properly embed a disk d_i in the tangle containing the single box so that T will intersect d_i in exactly two points p_1^i, p_2^i and $\partial d_i \subset \partial B_k$. Now consider the tangle (B_1, T_1). We can insert the disk d_1 below the second row of boxes and then insert a disk every four rows. Since the number of rows m is always odd, the number of disks is $\left[\frac{m-2}{4}\right] + 1$, where $[a]$ denotes the integer value of a. If the residue of $m \mod 4$ is 3, we will have a type A sub-tangle at the top and at the bottom. If $m \mod 4$ is 1, we will have a type A sub-tangle at the top and a type B at the bottom. If the number of rows is equal to or greater than 7 the sub-tangles in the middle will be of type C. Note that sub-tangles of type A and B appear only at the top and bottom of (B_k, T_k). In the case of the tangle (B_2, T_2) we will have a sub-tangle of type B at the top and if $m \mod 4 = 1$, we will have a type A
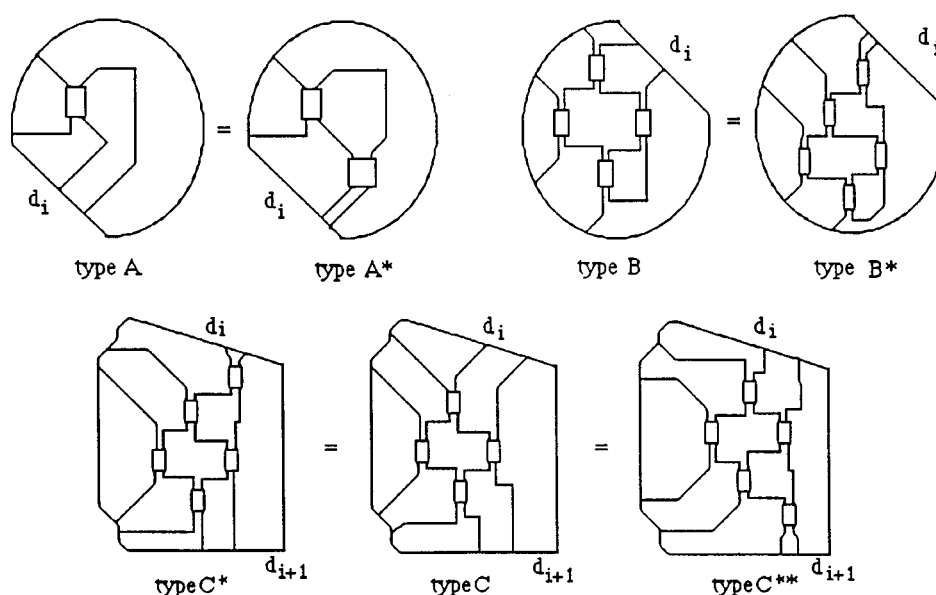

FIGURE 3

sub-tangle at the bottom. If $m \mod 4 = 3$, we will have a sub-tangle of type B at the bottom. The rest of the sub-tangles are of type C. Altogether we have $[\frac{m}{4}]$ disks in this case. Now the claim follows by an easy induction argument on m. $\square$

By the "configuration of the strings" we mean the linking pattern of the various strings, i.e., which strings link and in what order. The configuration of the strings in the sub-tangles of type A, B and C is determined by the parity of the crossing numbers $a_{i,j}$ that they contain (see Lemma 2.4 and Figure 3). Each sub-tangle of type B or C contains four such crossing numbers and is thus determined by four integers a_1, a_2, a_3, a_4. There are sixteen sub-tangles of type B, (C) denoted by $B(1) - B(16), (C(1) - C(16))$.

Lemma 2.6. *Let $(B, t_1, t_2) = (D^2, p_1, p_2) \times I$ be the trivial 2-tangle. Then there is a unique disk separating the two strings up to an isotopy fixing a neighborhood of the strings.*

Proof. Let D_1 be a disk in (B, t_1, t_2) which is (diameter of D^2) $\times I$. Let D_2 be a separating disk not isotopic to D_1 chosen so as to minimize the number of components of the intersection of all such disks with D_1. The intersection $D_1 \cap D_2$ cannot contain simple closed curves since an innermost such curve on D_2 would bound a 3-ball together with the disk on D_1. This 3-ball cannot contain the string since it does not intersect ∂B. By the irreducibility of B^3 we can isotope D_2 to reduce the number of components of $D_1 \cap D_2$ contradicting the choice of D_2. Hence we can assume that $D_1 \cap D_2$ is composed of arcs. Note that D_1 cuts (B, t_1, t_2) into two 3-balls B_1, B_2 each of which contains only one string. Let $D' \subset D_2$ be an outermost disk in D_2 cut off by an arc of $D_1 \cap D_2$. We may assume that D' is contained in, say, B_1. Now D' cuts B_1 into two 3-balls, one of which does not contain the string t_1. We can now isotope D_2 off that 3-ball to reduce the number of components of intersection of $D_1 \cap D_2$ by an isotopy fixing a neighborhood of both strings. Again this is in contradiction to the choice of D_2 and proves the claim. $\square$

3. ESSENTIAL TANGLES

In this section we prove a special case of Theorem 1.1 for knots or links admitting a 6-plat projection. Throughout this section we assume that all the conditions required in Theorem 1.1 are satisfied by all knots and links K.

Definition 3.1. An arc in one of the disks d_i will be called essential if it cannot be isotoped into ∂d_i by an isotopy fixing the points p_1, p_2. Otherwise, it will be called inessential.

Definition 3.2. An embedded disk $D \subset (B - T)$ will be called a compressing disk for the tangle (B, T) if $\partial D \subset \partial B - N(T)$ represents a nontrivial element in $\pi_1(\partial B - N(T))$.

Proposition 3.3. *Let $\mathcal{D}$ be a compressing disk for the tangle (B_k, T_k), $k = 1, 2$, chosen so that $|\mathcal{D} \cap \{\cup d_i\}|$, the number of components in the intersection $\mathcal{D} \cap \{\cup d_i\}$, is minimal. Then the set $\{\mathcal{D} \cap \{\cup d_i\}\}$ is not empty and contains no simple closed curves or inessential arcs.*

In order to prove the proposition we need the following lemma:

Lemma 3.4. *Every nontrivial simple closed curve σ on the boundary of a sub-tangle (B, T) of type A, B or C which misses the disks d_i is nontrivial in the*

sub-tangle. In particular, the boundary curves ∂d_i, of the disks d_i, in the tangles (B_k, T_k), $k = 1, 2$, are essential curves in $B_k - T_k$.

Proof. Suppose, in contradiction, that some nontrivial curve σ on the boundary of a sub-tangle (B, T) which misses the disks d_i bounds a disk Δ contained in the sub-tangle.

Case 1. The disk Δ cannot be contained in a sub-tangle (B, T) of type A: If the number of crossings a_1 in (B, T) is odd, the curve σ, being nontrivial, bounds a disk B^* on $\partial B - d_i$ containing at least one endpoint of at least one of the strings. Since a_1 is odd, the disk B^* cannot contain the other endpoint which is on d_i and hence Δ separates this string. This is a contradiction as $\Delta \cap T = \varnothing$. If a_1 is even, then Δ must separate the sub-tangle into two 3-balls each containing a string. Consider a small arc α on d_i connecting the two endpoints of the string t_1 which has both endpoints on d_i. Since $|a_1| > 1$, the simple closed curve $t_1 \cup \alpha$ is homologous to $[a_1/2] \neq 0 \in H_1(B - N(t_2))$, where t_2 is the other string in (B, T). Hence Δ cannot separate t_1 from t_2, a contradiction.

Case 2. The disk Δ cannot be contained in a sub-tangle (B, T) of type B: Consider the first five of sixteen sub-tangles of type B denoted by $B(1) - B(5)$ as indicated in Figure 4, where "o" indicates odd and "e" indicates even crossing number a_i.

In the sub-tangles of type $B(1)$–$B(5)$ the two endpoints p_1^1, p_2^1 on the disk d_i belong to the same string t_1. The disk Δ must therefore separate the string t_1 from the string t_2 or t_3 (or both). Since $\Delta \cap \text{int } d_i = \varnothing$, we can connect the endpoints of t_1 by an arc α on d_i pushed slightly into the interior of B. We denote these modified sub-tangles, which consist of a 3-ball B containing two properly embedded arcs and a simple closed curve $t_1 \cup \alpha$, by $B'(1)$–$B'(5)$ respectively. Therefore, if a disk Δ exists in a sub-tangle of type $B(1)$–$B(5)$, it must also exist in the modified sub-tangle of type $B'(1)$–$B'(5)$.

The curve $\sigma = \partial \Delta$ bounds a disk B^* on $\partial B - d_i$ and B^* must contain some endpoint of a string that Δ separates from t_1 and hence it must contain both endpoints of that string. If the disk B^* contains $\partial t_2(\partial t_3)$, we can connect the endpoints of the string $t_2(t_3)$ by an arc $\beta(\gamma)$ in B^*.

If we can separate t_2 from t_1 by a disk Δ, then we can also do this in the absence of t_3. So we can omit t_3 and after connecting the endpoints of t_1 and t_2 by arcs α and β, respectively, we obtain a 2-bridge link: $L(\alpha, \beta) = (t_1 \cup \alpha) \cup (t_2 \cup \beta)$. In cases $B'(1), B'(2)$ and $B'(5)$ the link $L(\alpha, \beta)$ is determined by the rational number $\alpha/\beta = [a_1, -a_3, a_4]$, where $[a_1, -a_3, a_4]$ is the continued fraction expansion. The separating disk Δ will be a half of a splitting sphere for the links. Since $|a_i| > 1$,
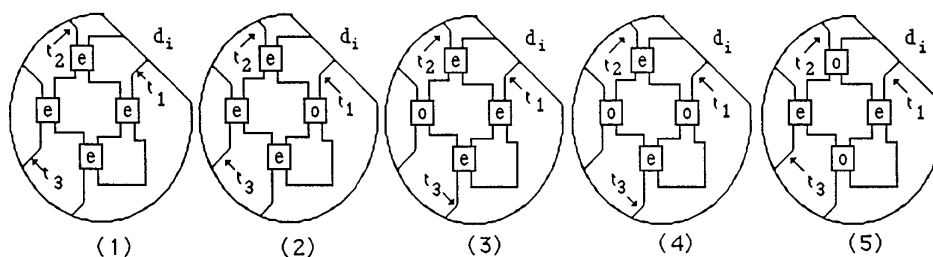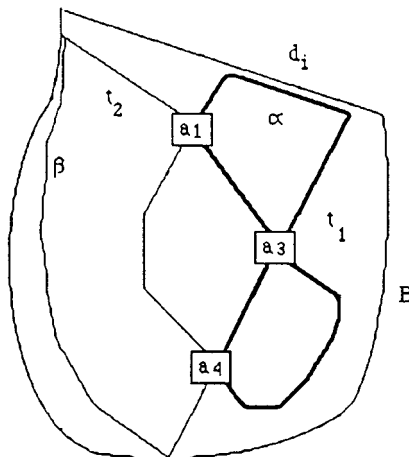

FIGURE 4

FIGURE 5

$i = 1, 3, 4$, the link $L(\alpha, \beta)$ is not a trivial link and by definition of a 2-bridge link it is not a split link. In cases $B'(3)$ and $B'(4)$ we obtain a torus link $L(2, a_1)$ which is also non-split since $|a_i| > 1$. Hence we cannot have a disk Δ separating t_2 from t_1 (see, for example, Figure 5 for case $B'(1)$).

Suppose now that we can separate t_3 from t_1 by a disk Δ, then we can do this in the absence of the string t_2. As before, in cases $B(3), B(4)$ and $B(5)$ we can omit the string t_2 and connect the endpoints of t_1 and t_3 by arcs α and γ, respectively, to obtain torus links $L = (t_1 \cup \alpha) \cup (t_3 \cup \gamma)$ of the form $L(2, a_4), L(2, a_4)$ and $L(2, a_2)$ respectively. These links are non-split since $|a_i| > 1$, so we cannot have a disk Δ separating t_3 from t_1 in these cases.

We still need to show that in cases $B(1)$ and $B(2)$ we cannot separate the string t_3 from the strings $t_1 \cup t_2$ by a disk Δ. The argument must be different since in the absence of t_2 we *can* separate t_3 from t_1 by a disk Δ.

If there is a disk Δ in one of $B'(1), B'(2)$ which splits off the unknotted arc t_3, then it also cuts $B'(i)$, $i = 1, 2$, into two 3-balls. One is a 3-ball $B^{\#}$ with an *unknotted* arc t_3 and the other $B^{\#\#}$ is a 3-ball containing the arc t_2 and the simple closed curve $t_1 \cup \alpha$. Hence there is an arc γ on the boundary of the $B^{\#}$ in the complement of the gluing disk Δ so that $\gamma \cup t_3$ bounds a disk in $B^{\#}$. This disk survives the gluing of $B^{\#}$ to $B^{\#\#}$ so the curve $\gamma \cup t_3 \subset \partial B - d_i$ bounds a disk in $B - N(t_1 \cup \alpha \cup t_2)$. Hence $[\gamma \cup t_3] = 1$ in $\pi_1(B - N(t_1 \cup \alpha \cup t_2))$.

The arc γ is a path on the meridional annulus $(\partial B - \partial t_2)$ with the two endpoints of t_3 removed. Hence we can omit one endpoint p of t_2 and consider γ as a path on the disk $\partial B - N(p)$ connecting the endpoints of t_3 in the complement of a third point p. Such a path, which determines an element in the 3-braid group, can wind alternately between the endpoints of t_3 and the remaining endpoint of t_2. Thus the contribution of the γ part of the simple closed curve $\gamma \cup t_3$ to the word $[\gamma \cup t_3]$ is v^m, where v is the generator of $\pi_1(B - N(t_1 \cup \alpha \cup t_2))$ corresponding to the foot of the string t_2 and $m \in \mathbb{Z}$, as indicated in Figure 6.

The t_3 part of the simple closed curve $\gamma \cup t_3$ contributes to $[\gamma \cup t_3]$ an element in $\pi(B - N(t_1 \cup \alpha \cup t_2))$ which depends on the linking of t_3 and $t_1 \cup \alpha \cup t_2$. In both $B'(1)$ and $B'(2)$ a_4 is even and hence $1 = [\gamma \cup t_3] = z^{k_2} v^m$, where $z = (uv)^{-k_4} v(uv)^{+k_4}$
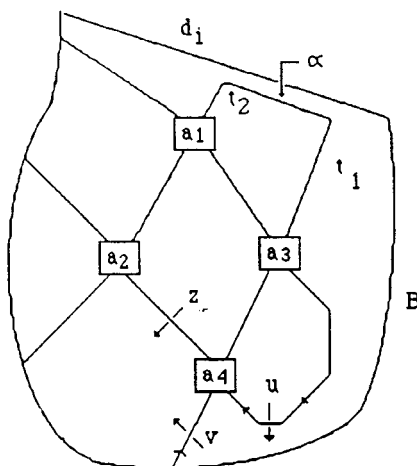
FIGURE 6

in case $B'(1)$, or $z = (u^{-1}v)^{-k_4}v(u^{-1}v)^{+k_4}$ in case $B'(2)$. If the word $z^{k_2}v^m$ is a relator in a group of a 2-bridge link, which abelianizes to $\mathbb{Z} \oplus \mathbb{Z}$, we must have $m = -k_2$.

Note that in each of $B'(1), B'(2)$ the fundamental group $\pi_1(B - N(t_1 \cup \alpha \cup t_2))$ is the group of a 2-bridge link $L(\alpha, \beta)$, $\alpha/\beta = [a_1, -a_3, a_4]$. Since $|a_i| > 1$, these links are never of the form $L(\alpha, 1)$ i.e., they are not torus links. Hence their complements $S^3 - N(L(\alpha, \beta)) = B - N(t_1 \cup \alpha \cup t_2)$ are hyperbolic.

Any 2-bridge knot or link group G has a presentation, with generators as indicated in Figure 6, as follows (see [BZ], p. 208):

$$G = \langle u, v | u^{-1}L^{-1}vL] \rangle \text{ where } L = u^{\varepsilon_1}v^{\varepsilon_2}\cdots u^{\varepsilon_{\alpha-2}}v^{\varepsilon_{\alpha-1}}$$

when $\alpha = 1 \mod 2$ i.e., $L(\alpha, \beta)$ is a knot and

$$G = \langle u, v | v^{-1}L^{-1}vL] \rangle \text{ where } L = u^{\varepsilon_1}v^{\varepsilon_2}\cdots v^{\varepsilon_{\alpha-2}}u^{\varepsilon_{\alpha-1}}$$

when $\alpha = 0 \mod 2$ i.e., $L(\alpha, \beta)$ is a link, where $\varepsilon_i = (-1)^{[i\beta/\alpha]}$ and $[a]$ is the integral part of a. Now set $a_i = 2k_i$ if a_i is even and if a_i is odd set $a_i = 2k_i \pm 1$, depending on whether a_i is positive or negative.

We need the following lemma:

Lemma 3.5. *The word $z^{k_2}v^{-k_2}$ is not a relation in $\pi_1(S^3 - N(L(\alpha, \beta)))$.*

Proof. Assume first that $|k_2| > 1$. In this case the subgroup $\langle v, z \rangle$ generated by v and z in $G = \pi_1(S^3 - N(L(\alpha, \beta)))$ is non-cyclic. If it was cyclic there would be an element $w \in G$ so that $v = w^r$ and $z = w^s$. This would contradict the fact that the images of both v and z in the abelianization of G are generators of a $\mathbb{Z}$ factor, unless $r = s$. If $r = s$, then $v = z$ which contradicts our assumption $|k_2| > 1$. Hence the group $\langle v, z \rangle$ has a nontrivial center and by Lemma VI.1.5 of [JS] the manifold $M = B - N(t_1 \cup \alpha \cup t_2)$ must contain a nontrivial Seifert characteristic variety. This is a contradiction since M is a hyperbolic manifold. (See also Theorem VI.4.1 of [Sh].)

If $|k_2| = 1$, then $v = (uv)^{-k_4}v(uv)^{+k_4}$ or $v = (u^{-1}v)^{-k_4}v(u^{-1}v)^{+k_4}$. This is the relation of a $(2, k_4)$-torus link group H. Since $\{u, v\}$ is a generating set for the

group G, it follows that G is a quotient of H. Recall that $H = \langle x, y | x^2 = y^{k_4} \rangle$. Since G is hyperbolic, the centralizer of H must be mapped to $\{e\} \in G$. It follows that G contains a quotient of $\langle x, y | x^2 = y^{k_4} \rangle / \text{Center} = \mathbb{Z}_2 * \mathbb{Z}_{k_4}$ and all elements of finite order must be in the kernel since G does not contain such elements. This would imply that the images of both x and y are in the kernel. However, the images of x and y generate the nontrivial group G. This is a contradiction and we cannot have a relation $z = v$ in G. □

Hence we cannot have a disk Δ separating t_3 from t_1 in sub-tangles of type $B'(1)$ and $B'(2)$. Consequently, we cannot have a disk Δ in sub-tangles of type $B(1)$–$B(5)$.

Consider now the remaining eleven sub-tangles of type B denoted by $B(6)$–$B(16)$ as indicated in Figures 10 and 11 and where $B(16)$ denotes the tangle where all a_i, $i = 1, \ldots, 4$, are odd. Note that in all the tangles $B(6)$–$B(16)$ the strings t_1 and t_2 have endpoints on the disk d_i so as before, since the disk Δ if it exists misses the disk d_i, we can connect the endpoints of t_1 and t_2 by an arc α to obtain one string $t = t_1 \cup \alpha \cup t_2$. We denote the new 2-tangles by $B'(6)$–$B'(16)$ and the existence of a disk Δ in $B(6)$–$B(16)$ implies the existence of Δ in $B'(6)$–$B'(16)$.

We can cut each of the sub-tangles $B'(6)$–$B'(16)$ by a properly embedded disk Δ_1 into two 2-tangles B_1 and B_2 separating the a_2 twists from the rest of the tangle. We can then further cut the 2-tangle B_2 by a properly embedded disk Δ_2 into two 2-tangles B_2^1 and B_2^2 separating the a_1 twists from the a_3 and a_4 twists as indicated in Figure 7. The tangles B_1 and B_2^1 are a_2 and a_1 twist tangles, respectively, in the terminology of [Wu 2] and the tangle B_2^2 is a rational tangle. All of B_1, B_2^1 and B_2^2 are atoroidal, i.e., the complement of the strings does not contain essential tori, since the complements of the strings are handlebodies. By Lemma 3.2 of [Wu 2] the tangle B_2 is atoroidal and a second application of the lemma for B_2 and B_1 shows that each of $B'(6)$–$B'(16)$ is atoroidal. Since $|a_{i,j}| > 1$, none of $B'(6)$–$B'(15)$ is a rational tangle since if we remove t_3 the arc t is knotted (i.e., there is no embedded disk D so that $\partial D = t \cup \alpha$, where α is an arc on ∂B). It now follows from Lemma 3.1 of [Wu 2] that $\partial B - N(t \cup t_3)$ is incompressible. The tangle $B'(16)$ does not contain a compressing disk by Lemma 3.3 of [Wu 2], since it is a sum of atoroidal tangles, neither of which is a 2-twist tangle (since in the case the a_i's are all odd). Therefore there is no compressing disk Δ in $B'(6)$–$B'(16)$ and hence there is none in $B(6)$–$B(16)$. This finishes the proof of Case 2.

Case 3. The disk Δ cannot be contained in a sub-tangle (B, T) of type C: In this case there are also sixteen different sub-tangles depending on the parity of the a_i's. Suppose that (B, T) is of type $C(j)$. If $\sigma = \partial \Delta$ is a non-trivial simple closed curve in the annulus $\partial B - ((d_i \cup d_{i+1})$ less ($\cup$ endpoints of the strings)), then either it is parallel to the core of the annulus (if the strings are not fixed) or it bounds a disk on ∂B.

Suppose first that σ is parallel to the core of the annulus. The curve σ separates ∂B into two disks D_i, D_{i+1} containing the disks d_i, d_{i+1} respectively. The disk D_i (D_{i+1}) must contain all endpoints of the strings meeting the disk d_i (d_{i+1}). Otherwise, the disk Δ will separate the endpoints of at least one of the strings, which clearly cannot happen. Thus σ will remain nontrivial in $\partial B'$-(strings), where B' is obtained from B by connecting the endpoints of the strings on d_{i+1} by a small arc and pushing this arc into the interior of B. We obtain a new tangle which is the same as $B(j)$ and the proof follows from that case.
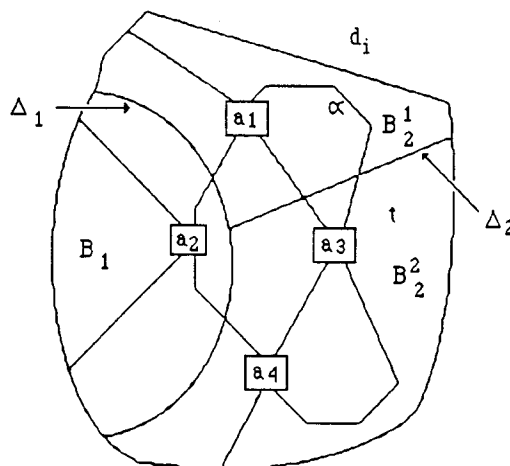
FIGURE 7

If σ bounds a disk B^* on the annulus, then we can connect the endpoints of the strings on the disk d_i by an arc in d_i and the curve σ will continue to be nontrivial in the new tangle, since both B^* and $\partial B - (B^* \cup d_i)$ contain endpoints of T. Again we obtain a new tangle which is the same as $B(j)$ and this completes the proof. $\square$

Proof of Proposition 3.3. If $\mathcal{D} \cap \{\cup d_j\}$ is empty, then $\mathcal{D}$ is contained on one of the sub-tangles of type A, B or C. However, if this happens, then $\partial \mathcal{D}$ is a nontrivial loop on ∂B which does not meet d_i and bounds a disk Δ in the sub-tangle. By Lemma 3.4 this cannot happen. If $\mathcal{D} \cap \{\cup d_j\}$ contains simple closed curves, then it must contain a simple closed curve γ bounding a sub-disk $d_i^* \subset d_i$, for some disk d_i. The disk d_i^* must contain both intersection points of $T \cap d_i$ since if $d_i^* \cap T$ is empty, we can reduce the number of components in the intersection $\mathcal{D} \cap \{\cup d_j\}$ violating the choice of $\mathcal{D}$ and if $d_i^* \cap T$ contains only one point, we obtain a 2-sphere intersecting K once. Hence γ is isotopic to ∂d_i and is compressible in contradiction to Lemma 3.4.

If any of the arcs of $\mathcal{D} \cap \{\cup d_j\}$ is inessential, we can replace $\mathcal{D}$ by a disk $\mathcal{D}^*$ with a smaller number of components in the intersection. This contradicts the choice of $\mathcal{D}$ and finishes the proof of Proposition 3.3. $\square$

Corollary 3.6. *An outermost disk $E \subset \mathcal{D}$, cut by $\mathcal{D} \cap \{\cup d_j\}$, must intersect $\{\cup d_j\}$ in a single arc $\rho \subset d_i$, for some $i \in \{1, \ldots, r\}$. The arc ρ must separate the two points of $T_k \cap d_i$, $k = 1, 2$, and hence E separates the corresponding strings in the corresponding sub-tangle of type A, B or C.*

Proof. By Proposition 3.3 every intersection arc of $\mathcal{D} \cap \{\cup d_j\}$ must be an essential arc in the disk d_j with the points p_1^j, p_2^j removed and since $E \subset \mathcal{D}$ is outermost it must intersect some d_i in a single arc ρ. $\square$

Theorem 3.7. *Let K be a knot in a 6-plat projection. If $m \geq 5$ and $|a_{i,j}| > 1$, for each i and j, then the tangles (B_1, T_1) and (B_2, T_2) are essential.*

Proof. Assume, in contradiction, that there is some compressing disk $\mathcal{D}$ for (B_k, T_k), $k = 1, 2$. We can choose a disk $\mathcal{D}$ so as to minimize the number of components
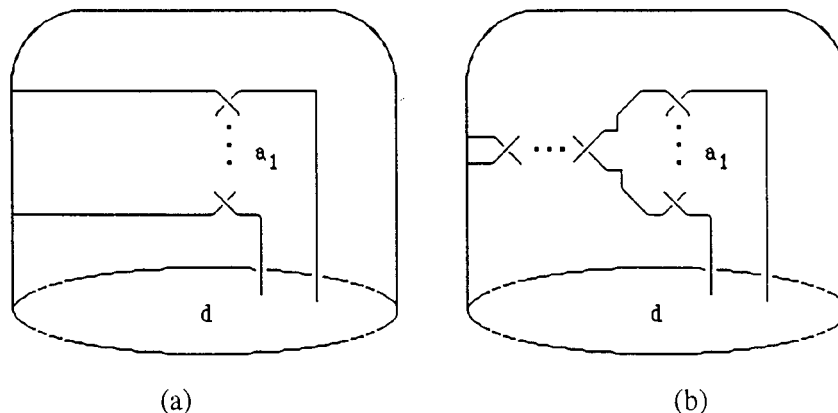
(a)   (b)

FIGURE 8

in the intersection of $\mathcal{D} \cap \{\cup d_j\}$. We may now assume, by Proposition 3.3, that $\mathcal{D} \cap \{\cup d_j\}$ contains no simple closed curves or inessential arcs. By Corollary 3.6 any outermost disk $E \subset \mathcal{D}$ must intersect a disk d_i in a single arc ρ separating the two points p_1^i, p_2^i and is contained in a single sub-tangle.

It is sufficient to consider only sub-tangles of type A or B. This is because the disk E cannot intersect both the top disk d_i and the bottom disk d_{i+1} in a sub-tangle of type C. Hence if a disk E like that exists in a sub-tangle of type C it will also exist in the sub-tangle after we connect the points p_1^i, p_2^i, on the disk not met by E, by a small arc on the disk and then push it slightly into the interior of the 3-ball. The resulting tangle, up to rotation, is a sub-tangle of type B.

Case 1. Assume that E is contained in a sub-tangle of type A. The crossing number a_1 must be odd, since if a_1 is even, then p_1^1, p_2^1 belong to the same string and cannot be separated by E. Assume that a_1 is odd. The sub-tangle of type A is isotopic to a tangle as in Figure 8(a) and after an isotopy (not fixing the strings) to a tangle as in Figure 8(b). The boundary of the disk E is a union of two arcs $\sigma \cup \rho$ where σ is an arc on the twice punctured disk $\partial B - d$. If σ is not an essential arc on $\partial B - d$, then we have a 2-sphere intersecting the link L once. Hence σ is an arc separating the two endpoints of the strings on $\partial B - d$. By Lemma 2.6 σ is isotopic by an isotopy of (B, T) twisting the endpoints of the strings in $\partial B - d$ about each other to the standard separating arc which we denote by σ'. The isotopy above would induce an ambient isotopy of the disk E to a new disk denoted by σ'. After the isotopy the tangle looks as indicated in Figure 8(b). The curve $\partial E' = \sigma' \cup \rho$ separates the pairs of endpoints of the two strings. We can now connect the endpoints of each respective string by an arc to obtain a 2-bridge link as indicated by the thick lines in Figure 9. Hence the existence of the disk E' implies that this 2-bridge link is a split link. This is a contradiction and hence a disk E does not exist.

Case 2. Assume that an outermost disk E is contained in a sub-tangle of type B. We must consider all sixteen possible cases.

Sub-case (a). Consider the sub-tangles of type $B(1)$–$B(5)$ as in Figure 4. In these sub-tangles the two endpoints p_1, p_2 on d_i belong to the same string. Since the disk E would have to separate them, such a disk cannot exist.
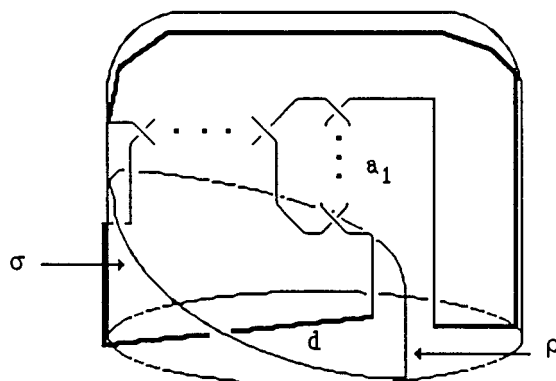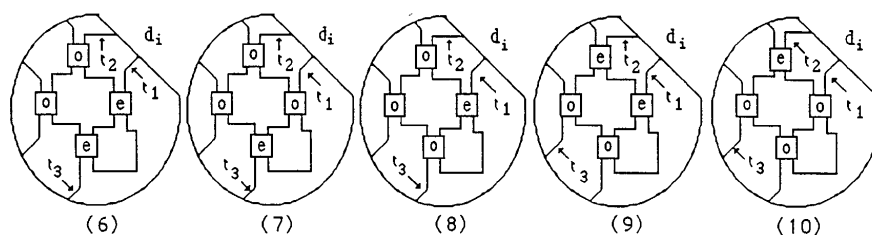
FIGURE 9

FIGURE 10

Sub-case (b). Consider the sub-tangles $B(6)$–$B(10)$ as in Figure 10.

The disk E if it exists must also separate the strings t_1 and t_2 in the absence of t_3. So omit the string t_3 from all sub-tangles. Now, in sub-tangles (6) and (7) we can twist the interior of the string t_1, fixing the endpoints of the strings, to get rid of the a_3 crossings. In sub-tangles (9) and (10) we can twist about the disk d_i, fixing its boundary, to also get rid of the a_3 crossings. In all cases, including sub-tangle (8), we obtain a sub-tangle of type A. The existence of a disk E in the sub-tangles of type (6)–(10) would imply the existence of such a disk in a sub-tangle of type A which is a contradiction. Hence we cannot have such an outermost disk in these sub-tangles.

Sub-case (c). Consider the sub-tangles $B(11)$–$B(15)$ as in Figure 11. As in sub-case (b) we can omit the t_3 string and the disk E, if it exists, must still separate the strings t_1 from t_2. By Lemma 3.8, which we prove below, the reduced word, represented by ∂d_i, in the free group $F(x,y) = \pi_1(H) = \pi_1(B - N(t_1 \cup t_2))$ has both generators occur at least twice if $|a_i| > 1$, $i = 1, 3, 4$. This implies that the intersection of the outermost disk E and d_i contains at least two arcs and rules out the existence of the disk E.

Lemma 3.8. *Let (B,T) be one of the sub-tangles of type $B(11)$–$B(15)$ with $|a_i| > 1$, $i = 1, 3, 4$, in each case. Let $d(x,y)$ be a reduced word in the fundamental group $\pi_1(B - N(t_1 \cup t_2)) = F(x,y)$ represented by the curve ∂d_i. Then $d(x,y)$ contains each of the generators x and y, as indicated in Figure 12 below, at least twice.*

Proof. Note that $d(x,y)$ is equal to $z^{-1}x$ as indicated in Figure 12. By checking all the different cases one sees that the word z is a conjugate of the generator $y^{\pm 1}$ by
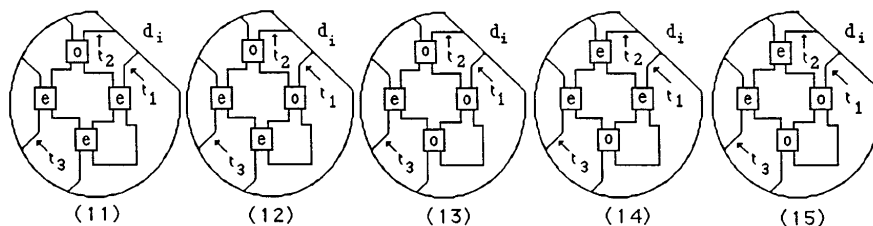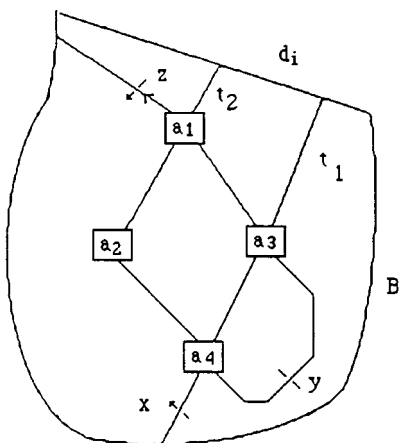
FIGURE 11

FIGURE 12

a word containing both generators x and y. Hence the number of occurrences of y in z is greater than or equal to 3 unless $z = y$. If $|a_i| > 1$, $i = 1, 3, 4$, clearly that is not the case.

The number of occurrences of x in z must be even and thus greater than or equal to 2. If x does not appear at the beginning or end of the word z, then the number of occurrences of x in $z^{-1}x$ must be greater or equal to 3. If x does appear at the end of z and xz^{-1} has only one occurrence of x, then, for some $m \in \mathbb{Z}$, z has the form $x^{-1}y^m y^{\pm 1} y^{-m} x = x^{-1} y^{\pm 1} x$. Again since $|a_i| > 1$, $i = 1, 3, 4$, it is clear that this is not the case.   □

Sub-case (d). Consider the sub-tangle of type $B(16)$. This case must be treated differently than the previous ones, since if we omit the t_3 string, we can find a disk E with the required properties. We add a 1-tangle (B, t') with a disk $d \subset \partial B$ to $B(16)$ along the disk d_i in such a way so that the endpoints of the string t' get identified with the endpoints of the strings t_1 and t_2 on d_i. The 1-tangle (B, t') contains a disk D intersecting t' in a single point and intersecting d in a single arc (see Figure 13). After the gluing of the two tangles we obtain a 2-tangle which is isotopic to $B'(16)$ as in the proof of Lemma 3.4

Assume now that the sub-tangle $B(16)$ contains an outermost disk $E \subset \mathcal{D}$ intersecting d_i in a single arc. We can arrange the gluing map between (B, t') and $B(16)$ so that the arc $E \cap d_i$ is identified with the arc $D \cap d$. Hence the existence of a compressing disk E in $B(16)$ implies the existence of an annulus $A = (D - N(t')) \cup E$ in
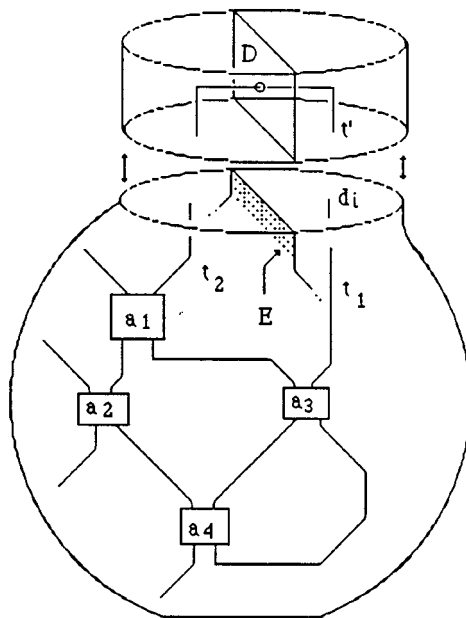
FIGURE 13

$B'(16)$. Note that A is incompressible, as one boundary component is a meridional curve, and not boundary parallel. Thus A is an essential annulus.

Consider now a decomposition of $B'(16)$ into sub-tangles B_1 and B_2 and then B_2 into B_2^1 and B_2^2 as in Figure 7. Recall that the tangles B_1 and B_2^1 are a_2 and a_1 twist tangles, respectively, in the terminology of [Wu 2] and that the tangle B_2^2 is a rational angle. All the sub-tangles B_1, B_2, B_2^1 and B_2^2 are atoroidal as the complements of a regular neighborhood of the strings are handlebodies. By Lemma 3.4 of [Wu 2] one of the three following cases must hold, up to exchanging between B_1 and B_2:

  (i) The tangle B_1 is Δ-annular.
 (ii) The tangle B_1 is a q-twist tangle, with q odd and the tangle B_2 is either a left torus tangle or a p-twist tangle with $|p| \geq 3$.
(iii) Both tangles B_i are wrapping tangles with the unknotted string of B_1 glued to the unknotted string of B_2.

For the definitions of Δ-annular, twist, torus and wrapping tangles see [Wu 2]. By Lemma 4.4 of [Wu 2] a nontrivial rational tangle is Δ-annular if and only if it is a $2n$-twist tangle with $|n| \geq 2$. A twist angle is, in particular, a rational tangle. Hence both B_1 and B_2^1 are not Δ-annular since both a_2 and a_1 are odd. This rules out the first case (i) for B_1. Since no arc of B_2 has both endpoints on the gluing disk Δ_1 between B_1 and B_2, the tangle B_2 is not a torus tangle. The tangle B_2 is clearly not a twist tangle, hence case (ii) is completely ruled out. Since both the strings in B_1 and B_2 are unknotted, neither B_1 and B_2 are wrapping tangles and hence case (iii) is completely ruled out. It remains to prove that B_2 is not Δ-annular.

If B_2 is Δ-annular, then B_2 contains an essential annulus by Lemma 3.5 of [Wu 2]. We can now apply Lemma 3.4 of [Wu 2] to $B_2 = B_2^1 + B_2^2$. Neither of B_2^1 or B_2^2 is Δ-annular since none of them is a $2n$-twist tangle. This rules out case (i). Since B_2^1 and B_2^2 are not twist tangles and no one of them is a torus tangle, case (ii) is ruled out. As both arcs in B_2^1 and B_2^2 are unknotted, they are not wrapping tangles and case (iii) is ruled out as well. Therefore, the tangle $B'(16)$ does not contain an essential annulus and hence we cannot have such a disk E in a subtangle of type $B(16)$. This finishes the proof of the theorem. $\square$

Corollary 3.9. *Let K be a knot or a link in S^3 admitting a 6-plat projection, $m \geq 5$. If $|a_{i,j}| > 1$, for each i and j, then the two planar meridional surfaces $S - N(K)$ are essential in $S^3 - N(K)$ and are not isotopic there.*

Proof. By Theorem 3.7 each of the two vertical 2-spheres S in the 6-plat knot K cuts the pair (S^3, K) into two essential sub-tangles $(B_1, T_1), (B_2, T_2)$ respectively. Hence $S^3 - N(K)$ contains two meridonal essential planar surfaces $S - N(K)$. The two planar surfaces are not isotopic since the tangles they divide (S^3, K) into are different. $\square$

Corollary 3.10. *Let K be a knot in S^3 admitting a 6-plat projection, $m \geq 5$. If $|a_{i,j}| > 1$, for each i and j, then K is a nontrivial knot.* $\square$

4. General $2n$-plats

In this section we will prove Theorem 1.1 for the general case. We consider knots or links $K \subset S^3$ in a $2n$-plat projection where $n \geq 3$ and m, the number of rows of crossings as in Definition 1.1, is odd and greater than or equal to 5. Let S be a vertical 2-sphere (see Definition 2.2) which intersects the knot K on the $n - 1$ (counted from the left) top and bottom bridges and then separates periodically one box of crossings in the first three rows and two boxes of crossings in the following row (counted from the right). See Figure 14.

The 2-sphere separates (S^3, K) into two tangles $(B_1, T_1), (B_2, T_2)$ and we denote by (B_1, T_1) the tangle containing the nth bridge. Note that if $n > 3$, then (B_1, T_1) is an essential tangle by Theorem 3.7 and if $n = 3$, then so is (B_2, T_2). Therefore, the goal of this section is to prove that with appropriate conditions the tangle (B_2, T_2) is essential when $n > 3$. Since the number of strings in the tangle (B_2, T_2) is always greater than or equal to two, it is sufficient to prove that ∂B_2 is incompressible in the complement of T_2 (i.e., we do not have to worry about boundary compressibility).

Let n be greater than or equal to 4. Then tangle (B_2, T_2) can be cut by a disk D into two sub-tangles (B_2^1, T_2^1) and (B_2^2, T_2^2) defined as follows:

The boundary of D is on S and the disk D cuts through the $(n - 2)$nd top and bottom bridges (counted from the left). In each row of boxes of crossings it cuts through the row one box further to the left than S as indicated in Figure 15 (a) for $m = 9$ and $n = 5$. The tangle (B_2, T_2) is the union of the tangles (B_2^1, T_2^1) and (B_2^2, T_2^2) along the disk D. Denote the "small" sub-tangle i.e., the one containing only one box of crossings from each row by (B_2^1, T_2^1) (see Figure 15 (b)).

Note that there are $m - 1$ disks $d_1, \ldots, d_{m-1}$ ($m \geq 5$ being the number of rows of boxes of crossings) properly embedded in (B_2^1, T_2^1). Each disk d_i cuts the strings
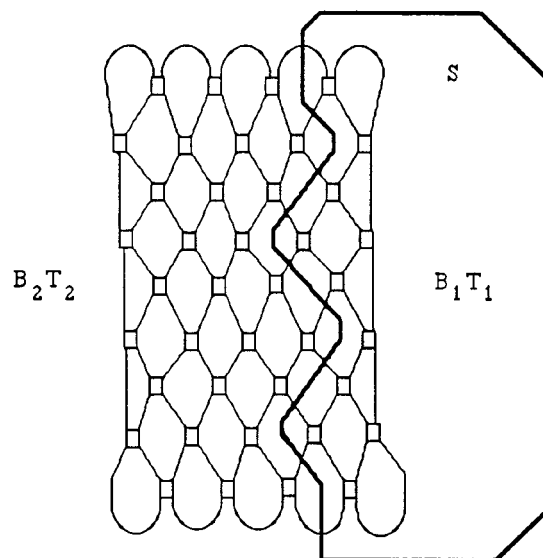
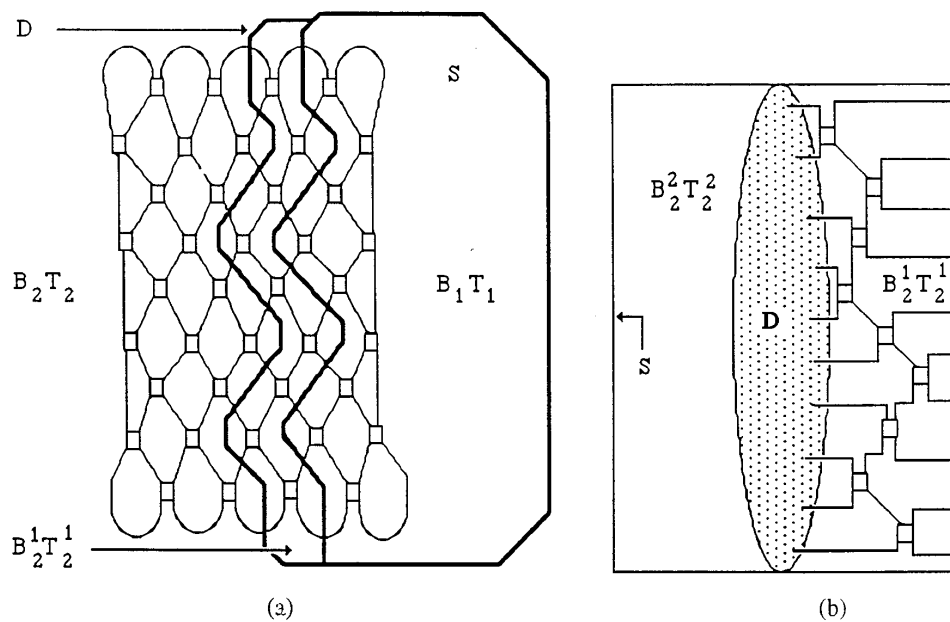FIGURE 14

(a)

(b)

FIGURE 15

T_2^1 in exactly one point p_i. The disks $d_1, \ldots, d_{m-1}$ separate the tangle (B_2^1, T_2^1) into m sub-tangles $(B^1, T^1), \ldots, (B^m, T^m)$ (indicated in Figure 16).
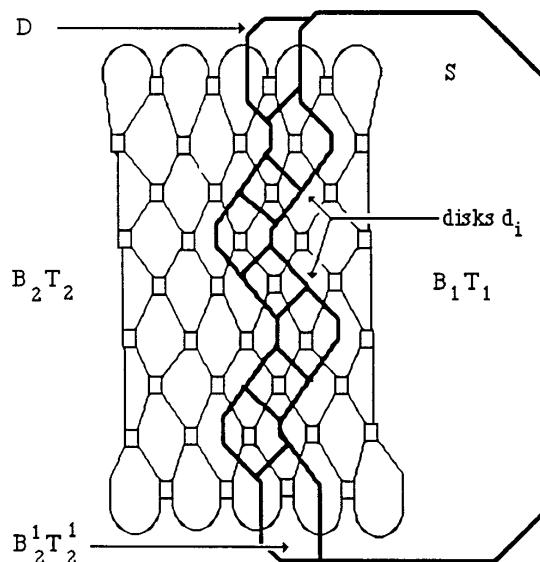
We have the following lemma:

FIGURE 16

Lemma 4.1. *Let K be a knot or link in S^3 in a $2n$-plat projection. Let S, D and $d_1, \ldots, d_{m-1}$ be as above. Then each of the tangles (B^j, T^j), $j = 1, \ldots, m$, is one of the following six types up to reflections:*
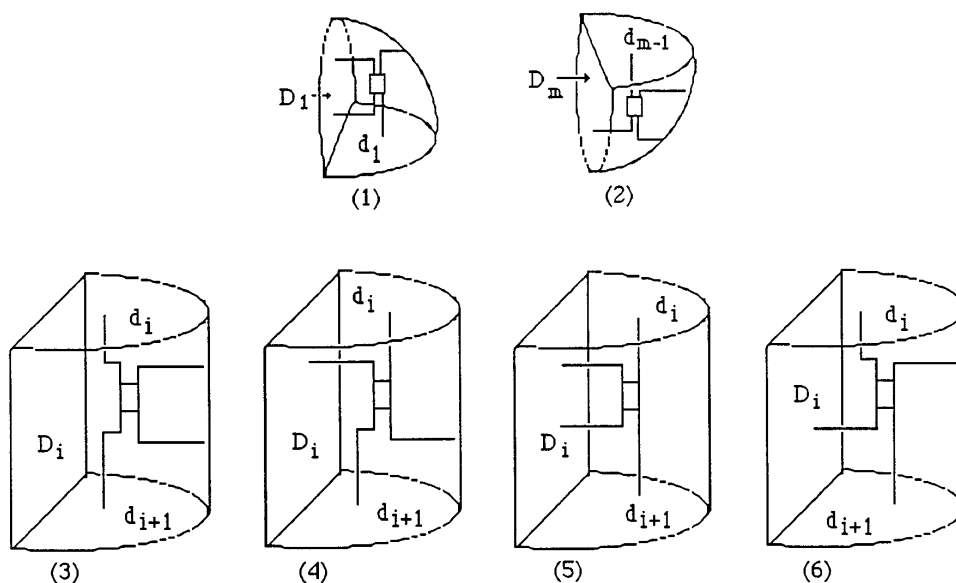

FIGURE 17

Proof. Consider the sub-tangles appearing at the "top" and "bottom" of (B_2^1, T_2^1), i.e., (B^1, T^1) and (B^m, T^m). From the definition of the vertical 2-sphere S and

the disk D it follows that $\partial(B^1, T^1)$ and $\partial(B^m, T^m)$ is the union of three disks, namely d_i, a sub-disk D_i of D and a sub-disk S_i of the disk $S^* = \partial(B_2^1, T_2^1) - D = S \cap (B_2^1, T_2^1)$, $i \in \{1, m\}$. So the type of tangle is determined by the configurations of the endpoints of the strings. We must always have one string on d_i, $i \in \{1, m\}$, so if we have two endpoints on D_i and one on S_i, we have a sub-tangle of type (1) and if we have one endpoint on D_i and two on S_i, we have a sub-tangle of type (2). Note that it follows from the definitions that the sub-tangle (B^1, T^1) is always of type (1) but (B^m, T^m) can be either type (1) or type (2), depending on m.

Consider now the sub-tangles appearing in the "middle" of (B_2^1, T_2^1), i.e., (B^i, T^i), $i = 2, \ldots, m-1$. From the definition of the vertical 2-sphere S and the disk D it follows that $\partial(B^i, T^i)$ is the union of four disks, namely d_i, d_{i+1} a sub-disk D_i of D and a sub-disk S_i of the disk $S^* = \partial(B_2^1, T_2^1) - D = S \cap (B_2^1, T_2^1)$, $i = 2, \ldots, m-1$. As before, the type of tangle is determined by the configurations of the endpoints of the strings. We must always have the endpoints of two strings on d_i and d_{i+1}, one emanating from the top of the box of crossings and the other from the bottom. So, if we have two endpoints on S_i and one on D_i, we have a sub-tangle of type (3). If we have two endpoints on D_i and none on S_i, we have a sub-tangle of type (5). If we have an endpoint of one of the top strings on D_i and an endpoint of one of the bottom strings S_i, we have a sub-tangle of type (4). Finally, if we have an endpoint of one of the bottom strings on D_i and an endpoint of one of the top strings on S_i, we have a sub-tangle of type (6). □

Lemma 4.2. *Let K be a knot or link in S^3 admitting a $2n$-plat projection with $m \geq 5$ and $|a_{i,j}| > 1$. Then there is no compressing disk Δ for the tangle (B_2^1, T_2^1) such that $\partial\Delta$ is contained in $S^* = \partial(B_2^1, T_2^1) - D$ or in D.*

Proof. Let Δ be a compressing disk for the tangle (B_2^1, T_2^1) so that $\Delta \cap D = \varnothing$. Choose such a disk which minimizes the number of components of intersection of $\Delta \cap \{\cup d_i\}$, $i = 1, \ldots, m-1$. Assume $\Delta \cap \{\cup d_i\}$ is empty. Note that $\partial\Delta$ is contained in $\partial(B_2^1, T_2^1) - D$ and bounds a disk B^* there. Cases (1), (4), (5) and (6) of Lemma 4.1 cannot occur since then B^* can contain only one or less endpoints of T and this implies that either $B^* \cup \Delta$ is a 2-sphere intersecting K in one point or $\partial\Delta$ is trivial. Cases (2) and (3) cannot occur since both cases are trivial 2-tangles and in such 2-tangles a compressing disk is unique and does not miss $\partial(B_2^1, T_2^1) - D$. We may conclude that for each i, $\Delta \cap d_i$ must be a collection of arcs and simple closed curves. If $\Delta \cap d_i$ contains any simple closed curve γ, then an innermost γ bounds a sub-disk d_i^* of d_i. The disk d_i^* cannot contain the point p_i on d_i since then we would have a 2-sphere $d_i^* \cap$ (disk bounded by γ on Δ) intersecting the knot or link K in a single point. However, if d_i^* does not contain the point p_i on d_i, then we can find a disk with a smaller number of components of intersection contrary to the choice of Δ. We conclude that $\Delta \cap \{\cup d_i\}$ is a collection of arcs cutting Δ into sub-disks.

Let E be an outermost sub-disk of Δ. The disk E is contained in a unique sub-tangle (B^i, T^i), $i = 1, \ldots, m$, which is of type (1)–(6), by Lemma 4.1. The boundary of E is a simple closed curve which consists of two arcs $\alpha \cup \delta$ where α is an arc on d_i and δ is an arc on S^*. Recall that we are assuming that $\partial E \cap D = \varnothing$.

Case 1. Assume that the outermost disk E is contained in a sub-tangle of type (1) or (2). The curve ∂E must separate $\partial(B^i, T^i)$, $i \in \{1, m\}$ into two disks each containing exactly two endpoints of the strings in (B^i, T^i). Otherwise, either the

intersection of Δ with $\cup d_i$ is not minimal or we get a 2-sphere intersecting the knot or link K in a single point. Furthermore, each pair of endpoints separated by E must, of course, belong to the same string. Now connect each pair of endpoints by an arc on their respective disks to obtain a two component link. This link is a $(2, a_{1,n-2})$ or a $(2, a_{m,n-2})$-torus link which is not split as $|a_{i,j}| > 1$. Hence a disk E cannot exist in sub-tangles of type (1) or (2).

Case 2. Assume that the outermost disk E is contained in a sub-tangle (B^i, T^i) of types (3)–(6). The curve ∂E cannot intersect both the disk d_i and d_{i+1} in the sub-tangle (B^i, T^i) and also must separate $\partial(B^i, T^i)$ into two disks each containing exactly two endpoints belonging to the same string, as in case (1). This rules out the possibility of (B^i, T^i) being of type (5) since the fact that two endpoints are contained in D requires that any such disk E must intersect both d_i and d_{i+1}. For the remaining cases we can connect each pair of endpoints by an arc on their respective disks to obtain a two component link as before. This link is a $(2, a_{i,n-1}), (2, a_{i,n-2})$ or $(2, a_{i,n-3})$-torus link depending on i, $i = 2, \ldots, m-1$, which is not split as $|a_{i,j}| > 1$. Hence a disk E cannot exist in sub-tangles of type (3), (4) or (6).

If we assume that $\partial\Delta$ is contained in D, a close observation shows that the same configurations of endpoints occur in the sub-disks of $\partial(B^i, T^i)$ separated by ∂E, as in the previous case. Therefore, we obtain a contradiction as before.   □

4.3. **Proof of Theorem 1.1.** Consider the vertical 2-sphere S separating (S^3, K) into two tangles (B_1, T_1) and (B_2, T_2). By Theorem 3.7 the tangle (B_1, T_1) is essential. So in order to finish the proof we need to show that the tangle (B_2, T_2) is essential for $n \geq 3$.

The proof will be by induction on n. For $n = 3$ the tangle (B_2, T_2) is essential by Theorem 3.7. Let n be greater than 3 and assume that the tangle (B_2, T_2) is essential for all knots or links K in a $2k$-plat projection with $m \geq 5$ and $|a_{i,j}| > 1$, for each i and j and all $k \leq n-1$. Now consider the tangle (B_2, T_2) for a knot or link K with a $2n$-plat projection satisfying the other conditions above.

Insert the disk D into the tangle (B_2, T_2) separating it into two sub-tangles (B_2^1, T_2^1) and (B_2^2, T_2^2). Assume that Δ is a compressing disk for (B_2, T_2). The curve $\partial\Delta$ is contained in $\partial B_2 - N(T_2)$ and, in fact, we can assume, by irreducibility, that $\partial\Delta \subset S^*$. Consider now such a disk Δ which minimizes the number of components of intersection with D. The intersection $\Delta \cap D$ must consist of essential simple closed curves in $\partial B_2 - N(T_2)$. Otherwise, we get a contradiction to the choice of Δ as intersection minimizing. Let γ be an innermost such curve on Δ which bounds a sub-disk Δ^* of Δ. The disk Δ^* cannot be contained in (B_2^2, T_2^2) since this would contradict the induction assumption that the tangle (B_2^2, T_2^2) is essential. On the other hand, it is not contained in (B_2^1, T_2^1) as $\gamma \subset D$ and this would contradict Lemma 4.2. Hence we cannot have such a disk Δ and the tangle (B_2, T_2) is essential. Note that the arguments work for all vertical 2-spheres by symmetry. So, in particular, all vertical 2-spheres are essential and there are $2 \cdot (n-2)$ of them: Two for each bridge of the projection, except the end bridges, depending on the two possible partitions of the second row of twists. These planar surfaces are nonisotopic since the tangles $(B_1, T_1), (B_2, T_2)$ are different for the two choices of 2-spheres.   □

Remark 4.3. Theorem 6.1 of [GR] uses Propositions 2.2.1 and 2.3.1 of [CGLS] in order to conclude the existence of a closed incompressible surface. Proposition 2.2.1 deals with the case that the essential meridional surface F in the manifold M is

nonplanar and Proposition 2.3.1 deals with the case where F is planar in which case we can conclude that the closed surface is a torus. However, the assumption in both propositions on F is that F is the surface with the minimal number of boundary components realizing the given boundary slope (see [CGLS], p. 267). The planar essential meridional surface $\partial B_1 - N(T_1)$, as proved in the proof of Theorem 1.1, is not necessarily minimal in that sense. Hence we cannot decide whether F is planar or not and cannot conclude anything directly as to the genus of the closed incompressible surface.

5. Consequences

In this section we will show that the condition $|a_{i,j}| > 1$ can be weakened. It is an interesting question, therefore, what are the exact conditions required for a knot or link complement to contain an essential meridional planar surface. We will also describe some consequences of Theorem 1.1 and its proof.

Lemma 5.1. *Let K be a knot or link in a $2n$-plat projection with $n \geq 3$ and $m \geq 5$. Assume that the following conditions hold:*
    1. $|a_{i,j}| = 1$ *or* 0, *for* $i = 2 \bmod 4$, $j = 2n$, *or*
    2. $|a_{i,j}| = 1$ *or* 0, *for* $i = 0 \bmod 4$, $j = 1$, *and*
    3. *all other* $a_{i,j}$ *satisfy* $|a_{i,j}| > 1$.
Then $S^3 - N(K)$ contains $n - 2$ planar nonisotopic essential meridional surfaces. When $j = 1$ in (1) and $j = n$ in (2), the other $n - 2$ planar meridional surfaces are essential.

Proof. We can cut K by two vertical 2-spheres, so that the rightmost and leftmost tangles are homeomorphic, respectively, to the tangles (B_1, T_1) and (B_2, T_2) we obtained in the 6-*plat* case. (In the 6-plat case one vertical 2-sphere will suffice.) By Lemma 2.4 and Remark 2.5 the tangles (B_1, T_1) and (B_2, T_2), each, can be cut further into sub-tangles of type A^*, B^*, C^* and C^{**} which are isotopic to sub-tangles of type A, B and C respectively. This isotopy twists each of the sub-tangles fixing the disks d_i and in the process unwinds exactly the crossings $a_{i,j}$ satisfying conditions (1) and (2). Hence the sub-tangles satisfy the conditions of Theorem 3.7 independently of the values of these $a_{i,j}$'s. It follows that the tangle components of K isotopic to (B_1, T_1) and (B_2, T_2) of the 6-plat case are essential. As in the proof of Theorem 1.1 this implies that the (B_1, T_1) and (B_2, T_2) components of K are essential. By considering all possible vertical 2-spheres which are "parallel" to the first two we see that $S^3 - N(K)$ contains $n - 2$ planar nonisotopic essential meridional surfaces. If we consider the case $j = 1$, in (1) and $j = n$, in (2), we obtain the other $n - 2$ planar nonisotopic essential meridional surfaces by a similar argument. $\quad\square$

The following definition appears in [GR]. A knot or link L in S^3 is an n-string composite if the pair (S^3, L) can be expressed as the union of two essential n-string tangles and a tangle composite if it is an n-string composite for some n. If K is not an n-string composite, then it is n-string prime.

Corollary 5.2. *Let K be a knot or link in a $2n$-plat, $n \geq 3$ projection. If $m \geq 5$, $|a_{i,j}| > 1$, for each i and j, then K is an $(m + 1)/2$-string composite in at least $2n - 4$ different ways.*

Proof. In each such knot or link K there are $2n - 4$ vertical 2-spheres and by the proof of Theorem 1.1 each such 2-sphere divides (S^3, K) into two essential $(m + 1)/2$-tangles.   □

Corollary 5.3. *Corollary* 1.2 *of* [GR], *that tunnel number one knots in* S^3 *are n-string prime for all n, does not hold for knots with tunnel number greater than or equal to two.*

Proof. For each $n \geq 3$ choose a knot K satisfying the conditions of Theorem 1.1 and so that g.c.d.$(a_{i,j}) \neq 1$. By the proof of Theorem 1.1 it is an n-string composite and by [LM] it has tunnel number $n - 1$.   □

Remark 5.4. A. Thompson shows, in [Th], that either a knot in S^3 in thin position is in bridge position or the knot complement contains a planar meridional incompressible surface (for both definitions see [Th] or [HK]). Based on this result D. Heath and T. Kobayashi developed a method to search for thin positions for knots. Theorem 1.1 implies that one cannot use both results to conclude that a knot in S^3 in thin position is in bridge position for all knots K which admit a $2n$-plat, $n \geq 3$ projection with $m \geq 5$, $|a_{i,j}| > 1$, for each i and j.

ACKNOWLEDGEMENTS

REFERENCES

[BZ]   G. Burde and H. Zieschang, Knots, De Gruyter Studies in Mathematics, 5, New York, 1985. MR **87b:**57004

[CGLS] M. Culler, C. Gordon, J. Luecke, and P. Shalen, *Dehn surgery on knots*, Ann. of Math. **125** (1987), 237–300. MR **88a:**57026

[CL]   D. Cooper and D. Long, *Derivative varieties and the pure braid group*, Amer. J. Math. **115** (1993), 137–160. MR **94b:**57003

[Fi]   E. Finkelstein, *Closed incompressible surfaces in closed braid complements*, J. Knot Theory Ramifications **7** (1998), 335–379. CMP 98:13

[GL]   C. McA. Gordon and J. Luecke, *Reducible manifolds and Dehn surgery*, Topology **35** (1996), 385–409. MR **97b:**57013

[GR]   C. McA. Gordon and A. Reid, *Tangle decompositions of tunnel number one knots and links*, J. Knots Theory Ramifications **4** (1995), 389–409. MR **96m:**57016

[He]   J. Hempel, 3-*manifolds*, Ann. of Math. Studies 86, Princeton University Press, Princeton, N.J., 1976. MR **54:**3702

[JS]   W. Jaco and P. Shalen, *Seifert fibered spaces in* 3-*manifolds*, Mem. Amer. Math. Soc. **21** (1979). MR **81c:**57010

[HK]   D. Heath and T. Kobayashi, *A search method for a thin position of a link*, preprint.

[LM]   M. Lustig and Y. Moriah, *Generalized Montesinos knots, tunnels and* $\mathcal{N}$-*torsion*, Math. Ann. **295** (1993), 167–189. MR **94b:**57011

[LP]   M. T. Lozano and J. H. Przytycki, *Incompressible surfaces in the exterior of a closed* 3-*braid*, Math. Proc. Cambridge Philos. Soc. **98** (1985), 275–299. MR **87a:**57013

[Ly]   H. Lyon, *Incompressible surfaces in knot spaces*, Trans. Amer. Math. Soc. **157** (1971), 53–62. MR **43:**1169

[Me]   W. Mensco, *Closed incompressible surfaces in alternating knot and link complements*, Topology **23** (1984), 37–44. MR **86b:**57004

[Oe]   U. Oertel, *Closed incompressible surfaces in complements of star links*, Pacific J. Math. **111** (1984), 209–230. MR **85j:**57008

[Sh] H. Short, *Some closed incompressible surfaces in knot complements which survive surgery*, London Math. Soc. Lecture Notes Ser., **95** (1985), 179–194. MR **88d:**57006

[Sw] G. A. Swarup, *On incompressible surfaces in the complements of knots*, J. Indian Math. Soc. **37** (1973), 9–24. MR **50:**14757

[Th] A. Thompson, *Thin position and bridge number for knots in the 3-sphere*, Topology **36** (1997), 505–507. MR **97m:**57013

[Wu 1] Y. Q. Wu, *Incompressibility of surfaces in surgered 3-manifolds*, Topology **31** (1992), 271–279. MR **94e:**57027

[Wu 2] Y. Q. Wu, *The classification of nonsimple algebraic tangles*, Math. Ann. **304** (1996), 457–480. MR **97b:**57010

Department of Mathematics, (CUNY) Hunter College, New York, New York 10021
E-mail address: efinkels@shiva.hunter.cuny.edu

Department of Mathematics, Technion, Haifa 32000, Israel
E-mail address: ymoriah@techunix.technion.ac.il